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MATHEMATICAL INFINITY AND THE DIFFERENTIAL.

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Mathematics, as defined by the great mathematician, Benjamin Pierce, is the science which draws necessary conclusions. In its broadest sense, it deals with conceptions from which necessary conclusions are drawn. A mathematical conception is any conception which, by means of a finite number of specified elements, is precisely and completely defined and determined. To denote the dependence of a mathematical conception on its elements, the word "manifoldness," introduced by Riemann, has been recently adopted. Manifoldness may be looked upon as the genus, and function, as the species. This conception reaches down to the very foundation of mathematical concepts and principles. It is the central idea from which the whole field and range of the mathematical sciences may be surveyed. Time, space, and numbers are included in the notion, manifoldness.

Manifoldness may be defined according to Dr. Cantor as being in general every *muchness* or complexity which may be conceived as a unit, or a number of objects, conceptions, or elements which are united in one law or system.

Manifoldness may be devided into discrete and continuous. Proceeding with the conception of whole numbers as it is obtained by counting and extending the same by means of the divisibility of numbers so as to include the conception of the rational system of numbers, we have one of the elements which enter into the conception of a discrete manifoldness. The irrational system of numbers is included in the conception of continuous manifoldness. This must not be considered as an inherent division, for it is well to note here that in the higher analysis, in one instance and for one purpose, a conception may be considered as a discrete manifoldness and for another purpose as a continuous manifoldness.

The three laws of operation, i. e., the law of commutation, of association, and of distribution, hold good in all forms of calculation, whether discrets or continuous manifoldness. From these laws, the four processes, addition, subtraction, multiplication, and division are derived.

Number, in all its forms, whether finite or infinite, rational or irrational, constant or variable, continuous or discontinuous, is included as one of the elements of manifoldness.

We will now consider number with special reference to its limits, infinity and zero, by the introduction of the conception of variability, of continuity, and of the differential.

By means of an unlimited continuous series of rational numbers, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\ldots\ldots \alpha_n$, whose terms have the property that there be given to every number δ, however small, a place n, from which the difference of all succeeding numbers remains smaller than δ, we define a definite number which is called a limit of this series. The creation of this conception admits of a comparison of rational numbers with respect to their magnitude. If all the numbers of the series differ after the place α_n by less than the number δ, then the limit is a number which lies between $\alpha_n - \delta$ and $\alpha_n + \delta$, which, because δ may be chosen as small as we please, can be expressed by a rational number as near as we please.

The totality of all numbers of an interval, for example, 0 to 1, consists not only of all numbers between 0 and 1, but of the totality of all numbers which may be interpolated between the limiting values of the defined series of numbers. This totality we dominate the aggregate or inclusive of the continuous series of numbers.

It is apparent that the conception of a limit of variability and of continuity have their root in irrationality. The two conceptions attached to a limit are in their nature entirely different. In the first instance, a limit may be defined as a limit of a variable, a limitless increasing or decreasing; in the second instance a limit means that which exceeds all limits of measurable number, either because it possesses no magnitude or because the amount or extent would not be exhausted by means of all the series of all numbers though they were being perfected. In the first case, we deal with variable numbers ; in the second case with the conception of the absolute value of the numbers derived from the formation of zero and the conception of infinity.

Zero and infinity are the limits of the natural series of numbers. They are derived in the same manner as the rational series of numbers. Infinity is the result of unlimited addition of unity or other positive numbers, the unlimited multiplication of whole numbers except unity. Zero is derived from the subtraction of two equal numbers. These are the fundamental conceptions of zero and infinity as derived in the lower analysis. It is evident from the different ways in which each of these symbols are derived that they have different meanings attached to them. We may note here that every problem carries inherently with it its solution. The meaning of every symbol depends upon its origin, deriv-

ation and relation. In different problems they may have different meanings. Symbols of quantity, like words, have different definitions, and these are to be determined according to the nature of the problem and their relation to other symbols.

In the higher analysis, the conceptions of infinity and zero present themselves more systematically in the developement of infinite series, infinite products, infinite continued fractions, etc. An infinite number is defined as a variable number, whose absolute value is conceived as being in an unlimited state of increasing or decreasing. In the first instance it is called infinitely large, in the second, infinitely small. The addition of a number of infinitely large or infinitely small numbers will produce an infinitely large or small number. The difference between two infinitely large or infinitely small numbers, where either or both are equal, is zero. However, if they are not equal, the difference can never be a finite number, but must always be an infinite number; otherwise an infinite number would be increased or decreased by a finite number, which is without meaning.

The addition and subtraction of infinite numbers can never produce anything else than infinite numbers or, in a particular case, zero. Again the multiplication or division of an infinite number by a finite number or by infinity will produce like results, i. e., it may be merely an indicated operation, not a completed operation. For instance,

$$2\times\infty=2\,\infty\,;\; n\times\infty=n\,\infty\,;\; \infty\times\infty=\infty^2,\ \&c.$$

It is apparent that the unlimited number of changes which may be thought of under the conception of infinity as defined here are extraordinarily manifold.

If we conceive an infinite number to grow so that it is continuously twice as large as any other infinite number, then the first is derived from the second by multiplying by two or the second by dividing by two. Multiplying an infinite number by another gives us infinity of a higher power or dividing gives us infinity of a lower power. Every change in value of a variable suggests an increment.

There are two kinds of conceptions associated with increments: the one is that the absolute value of the increment is capable of divisibility. The conditions, however, of which are such that it cannot be conceived smaller. The other is that the absolute value is incapable of divisibility. In the first instance the increments are of such a nature that the variables must stand in a certain relation to one another and if this takes place they are known in higher mathematics as differentials; those of the second kind are of that nature that they do not stand in any relation to one another; these may be called absolute elements of quantity.

Thus, if we pass from one interval of value of a variable to another, there lies between the two a difference which must be considered as possessing quantity, but does not possess the capability of divisibility and this difference in in-

crements is an element of quantity. Increments and differentials are not identical. The former are vested with quantity, while the latter are vested with quality, i. e., they are formal in their nature.

Milwaukee, Wisconsin, September, 1896.

A PROBLEM IN ASTRONOMY.

By G. B. M. ZERR, A. M., Ph. D., Texarkana, Arkansas-Texas.

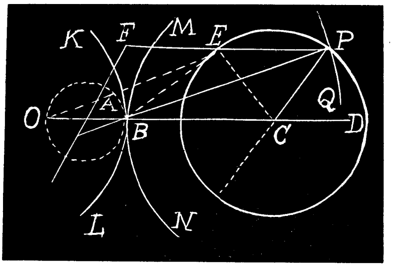
To find the Distance from the Earth to the Sun knowing the distance from an Inferior planet to the Sun supposing the planets to describe circles around the Sun.

Let P be a point on the epicyclic curve PQ, OC the radius of the deferent, CP the radius of the epicycle. Let $CO : CB = n : 1$. $\therefore CB = \frac{CO}{n}$.

Then $BO = CO - CB = CO - \frac{CO}{n} = CO(1 - \frac{1}{n})$.

Now the angular velocity$=\frac{\text{transverse velocity}}{\text{radius vector}}$.

The transverse velocity of E is represented by EA in magnitude, and in direction by BA. Let the linear velocity of the mean point (C) be V, the linear velocity of the moving point in the epicycle is



$$nV.\frac{PC}{CO}.$$

$\therefore$ The transverse velocity$=$

$$\frac{nV}{CO}.EB\cos BEO;$$

$$\text{but } \cos BEO = \frac{(BE)^2 + (EO)^2 - (BO)^2}{2BE.EO}.$$

$\therefore$ Transverse velocity $=\frac{nV}{CO}.\frac{BE^2 + EO^2 - BO^2}{2EO}$. Also radius vector$=EO$.

$$\therefore \text{Angular velocity} = \frac{nV}{CO}.\frac{BE^2 + EO^2 - BO^2}{2EO^2} \quad \ldots\ldots\ldots\ldots (A).$$